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Office Hour: Send me an email first, then we will arrange a meeting (if you need it).
Tutorial Arrangement:

- (1330 - 1355/ 15:30 - 15:55): Problems.
- (1355 - 1415/ 15:55 - 16:15): Class exercises.
- (1415 - 1430/ 16:15 - 16:30): Submission of Class Exercise via Gradescope.
- (1430 - 1530/ 16:30 - 17:30): Late submission period.

1 Line Integrals of Functions

1.1 Line Integrals of Functions over C^1 curves

Let C be a C^1 parametric curve and f be a function defined on C. Then the line integral of f over C is defined to be

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'|(t)\,dt.$$

Example 1:

Evaluate the line integral

$$\int_C f\,ds$$

where $f(x,y,z) = 2xy + \sqrt{z}$ and C is part of a helix, i.e., $\mathbf{r}(t) = (\cos t, \sin t, t)$ for $t \in [0,\pi]$.

Solution:
Given $\mathbf{r}(t) = (\cos t, \sin t, t)$, we have $\mathbf{r}'(t) = (-\sin t, \cos t, 1)$, hence

$$|\mathbf{r}'|(t) = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}.$$

Moreover,

$$f(\mathbf{r}(t)) = 2(\cos t)(\sin t) + \sqrt{t} = \sin 2t + \sqrt{t}.$$

Therefore,

$$\int_C f\,ds = \int_0^\pi (\sin 2t + \sqrt{t})\sqrt{2}\,dt = \frac{2\sqrt{2}}{3}\pi^{3/2}$$

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Example 2:

Evaluate $\int_C x\,ds$, where C is the parabolic curve $\mathbf{r}(t) = (t, t^2)$ for $t \in [0,2]$.

Solution:
Given $\mathbf{r}(t) = (t,t^2)$, then $\mathbf{r}'(t) = (1, 2t)$, hence

$$|\mathbf{r}'|(t) = \sqrt{1+4t^2}.$$

Moreover, in this case, $f(x, y) = x$, so

$$x = t.$$

Therefore,

$$\int_C f\ ds = \int_0^2 t\sqrt{1 + 4t^2}\ dt$$

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1.2 Line Integrals over Piecewise C^1 curves

If C is a piecewise C^1 curve joined by segments of C^1 curves $C_1, ..., C_k$ from end to end. Then the line integral of f over C is the sum of the line integrals over each C_i, i.e.,

$$\int_C f\ ds = \sum_{i=1}^{n} \int_{C_i} f\ ds$$

Example 3:

Evaluate $\int_{C_1 \cup C_2} \sqrt{x + 2y}\ ds$, where C_1 is the line segment from $(0, 0)$ to $(1, 0)$ and C_2 is the line segment from $(1, 0)$ to $(1, 2)$.

Solution:
The line segment C_1 is given by $(1 - t)(0, 0) + t(1, 0) = (t, 0)$ for $t \in [0, 1]$; while the C_2 is given by $(1 - t)(1, 0) + t(1, 2) = (1 - t + t, 2t) = (1, 2t)$ for $t \in [0, 1]$.

Let $\mathbf{r}_1(t)$ denotes the parametrization of C_1, then $\mathbf{r}_1'(t) = (1, 0)$ and hence $|\mathbf{r}_1'|(t) = 1$ for $t \in [0, 1]$.
Let $\mathbf{r}_2(t)$ denotes the parametrization of C_2, then $\mathbf{r}_2'(t) = (0, 2)$ and hence $|\mathbf{r}_1'|(t) = 2$ for $t \in [0, 1]$.

Then

$$\int_C \sqrt{x + 2y}\ ds = \int_{C_1} \sqrt{x + 2y}\ ds + \int_{C_2} \sqrt{x + 2y}\ ds,$$

where

$$\int_{C_1} \sqrt{x + 2y}\ ds = \int_0^1 \sqrt{t} \cdot 1\ dt$$

and

$$\int_{C_2} \sqrt{x + 2y}\ ds = \int_0^1 \sqrt{1 + 4t} \cdot 2\ dt$$